

Super quantum probabilities and three-slit experiments - Wright's pentagon state and the Popescu-Rohrlich box require third-order interference

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Abstract. Quantum probabilities differ from classical ones in many ways, e.g., by violating the well-known Bell and CHSH inequalities or another simple inequality due to R. Wright. The latter one has recently regained attention because of its equivalence to a novel noncontextual inequality by Klyachko et al. On the other hand, quantum probabilities still obey many limitations which need not hold any more in more general probabilistic theories (super quantum probabilities). Wright, Popescu and Rohrlich identified states which are included in such theories, but impossible in quantum mechanics, and they showed this using its Hilbert space formalism. Recently, Fritz et al. and Cabello detected that the impossibility of these states can be derived from very general principles (local orthogonality and global exclusive disjunction, respectively) without using Hilbert space techniques. In the paper, an alternative derivation from rather different physical principles will be presented. These are a reasonable calculus of conditional probability (i.e., a model for the quantum measurement process) and the absence of third-order interference. The concept of third-order interference was introduced by Sorkin who also recognized its impossibility in quantum mechanics.

1. Introduction

Quantum probabilities differ from classical ones in many ways, e.g., by violating the well-known Bell and CHSH inequalities [4, 8] or another simple inequality due to R. Wright [19]. The latter one has recently regained attention because of its equivalence to a novel noncontextual inequality by Klyachko et al. [1, 2, 6, 7, 11, 12].

On the other hand, quantum probabilities still obey many limitations which need not hold any more in more general probabilistic theories (super quantum probabilities). The study of these limitations as well as of the differences from classical probabilities is a current subject of research in the information theoretic and probabilistic foundations of quantum mechanics.

A very general probabilistic theory is provided by the quantum logics [5] with a sufficiently rich state space, but most of them do not allow for a reasonable calculus of conditional probability and thus lack in a model for the quantum measurement process.

However, those ones which entail a reasonable calculus of conditional probability yield a very fertile mathematical structure [13]; they shall be called UCP quantum logics and provide the probabilistic framework in this paper.

Wright [19] as well as Popescu and Rohrlich [15] identified limitations of quantum mechanics in the form of states which are included in some general probabilistic theories, but impossible in quantum mechanics, and they showed this using its Hilbert space formalism. Recently, Fritz et al. [9] and Cabello [7] detected that the impossibility of these states can be derived from very general principles (local orthogonality and global exclusive disjunction, respectively) without using Hilbert space techniques. An alternative derivation from a rather different physical principle will be presented in this paper. This is the absence of third-order interference. The concept of third-order interference was introduced by Sorkin who also recognized that third-order interference is ruled out by quantum mechanics [17]. His concept was adapted to conditional probabilities by Barnum, Emerson and Ududec [3].

Sections 2 and 3 briefly sketch the results from [13] concerning the quantum logics with a reasonable calculus of conditional probability and Sorkin's concept of third-order interference. In section 4, Wright's pentagon state is considered and it is shown that it requires third-order interference. This result is then applied to the Popescu-Rohrlich box in section 5.

2. The calculus of conditional probability

In quantum mechanics, the measurable quantities of a physical system are represented by observables. Most simple are those observables where only the two discrete values 0 and 1 are possible as measurement outcome; they are called *events* (or *propositions*) and are elements of a mathematical structure called *quantum logic* [5]. Quantum mechanics uses a very special type of quantum logic; it consists of the self-adjoint projections on a Hilbert space or, more generally, in a von Neumann algebra.

An abstractly defined quantum logic E contains two specific elements 0 and \mathbb{I} and possesses an *orthogonality relation* \perp , an *orthocomplementation* $E \ni e \rightarrow e' \in E$ and a *partial sum operation* $+$ which is defined only for orthogonal events. The interpretation of this mathematical terminology is as follows: orthogonal events are exclusive, e' is the negation of e , and $e + f$ is the disjunction of the two exclusive events e and f .

The states on a quantum logic are the analogue of the probability measures in classical probability theory, and conditional probabilities can be defined similar to their classical prototype [13]. A *state* μ allocates the probability $\mu(f) \in [0, 1]$ to each event f , is additive for orthogonal events, and $\mu(\mathbb{I}) = 1$. The *conditional probability* of an event f under another event e is the updated probability after the outcome of a first measurement has been the event e ; it is denoted by $\mu(f \mid e)$. Mathematically, it is defined by the conditions that the map $E \ni f \rightarrow \mu(f \mid e)$ is a state on E and that the identity $\mu(f \mid e) = \mu(f)/\mu(e)$ holds for all events $f \in E$ with $f \perp e'$. It must be assumed that $\mu(e) \neq 0$.

However, among the abstractly defined quantum logics, there are many where no states or no conditional probabilities exist, or where the conditional probabilities are ambiguous. Therefore, only those quantum logics where sufficiently many states and unique conditional probabilities exist can be considered a satisfying framework for general probabilistic theories. They shall be called *UCP quantum logics* in this paper and have been studied in Ref. [13]. Some of the results will be needed in this paper and shall now be sketched briefly.

A UCP quantum logic E generates an order-unit space A (partially ordered real linear space with a specific norm; see [10]) and can be embedded in its unit interval $[0, \mathbb{I}] := \{a \in A : 0 \leq a \leq \mathbb{I}\}$; \mathbb{I} becomes the order-unit, and $e' = \mathbb{I} - e$ for $e \in E$. Each state μ on E has a unique positive linear extension on A which is again denoted by μ [13].

For each event e in E , there is a positive linear map $U_e : A \rightarrow A$ with the following properties: $\mu(f \mid e) \mu(e) = \mu(U_e f)$ for all $f \in E$ and all states μ , $\mu(U_e x) = \mu(x)$ for all $x \in A$ and any state μ with $\mu(e) = 1$, $U_e^2 = U_e$, $e = U_e e = U_e \mathbb{I}$ and $0 = U_e f$ for $e \perp f$, $f = U_e f$ for $e' \perp f$. In quantum mechanics, $U_e x$ is the operator product $ex e$, which reveals the link to the quantum measurement process.

A linear map T_e can now be defined for each $e \in E$ by $T_e(x) := \frac{1}{2}(x + U_e x - U_{e'} x)$, $x \in A$. The properties of the maps U_e above imply the following properties for these maps: $\mu(T_e x) = \mu(x)$ for all $x \in A$ and any state μ with $\mu(e) = 1$, $e = T_e e = T_e \mathbb{I}$, and $0 = T_e f$ for $e \perp f$. In quantum mechanics, $T_e x$ is the Jordan product $e \circ x = (ex + xe)/2$.

These maps T_e ($e \in E$) and their properties will play a significant role in the proof of the theorem in section 3. Moreover, in [13], an interesting link between them and Sorkin's concept of third-order interference has been established, which shall be considered in the next section.

3. Sorkin's third-order interference

Sorkin [17] introduced the following mathematical term I_3 for a triple of pairwise orthogonal events e_1, e_2 and e_3 , a further event f and a state μ :

$$\begin{aligned} I_3 := & \mu(f \mid e_1 + e_2 + e_3) \mu(e_1 + e_2 + e_3) - \mu(f \mid e_1 + e_2) \mu(e_1 + e_2) \\ & - \mu(f \mid e_1 + e_3) \mu(e_1 + e_3) - \mu(f \mid e_2 + e_3) \mu(e_2 + e_3) \\ & + \mu(f \mid e_1) \mu(e_1) + \mu(f \mid e_2) \mu(e_2) + \mu(f \mid e_3) \mu(e_3) \end{aligned}$$

He recognized that $I_3 = 0$ is universally valid in quantum mechanics. His original definition refers to probability measures on 'sets of histories'. Using conditional probabilities, it gets the above shape, which was seen by Ududec, Barnum and Emerson [3].

For the three-slit set-up considered by Sorkin, the identity $I_3 = 0$ means that the interference pattern observed with three open slits is a simple combination of the patterns observed in the six different cases when only one or two of the three slits are open. This could be confirmed in a recent experimental test with photons [16].

The new type of interference which is present whenever $I_3 \neq 0$ holds is called *third-order interference*. In Ref. [13], it has been shown that a UCP quantum logic E rules

out third-order interference if and only if the identity $T_{e+f}x = T_ex + T_fx$ holds for all orthogonal event pairs e and f in E and all x in A . Mathematically, this orthogonal additivity of T_e in e is a lot easier to handle than the equivalent identity $I_3 \equiv 0$ with the above definition of the rather intricate term I_3 which, however, may be more meaningful physically. The orthogonal additivity of T_e in E will play a central role in the derivation of the result in the next section.

4. Wright's pentagon state

Consider a state μ , five events e_1, \dots, e_5 , the sum of their probabilities $\sum \mu(e_k)$, and assume $e_1 \perp e_2$, $e_2 \perp e_3$, $e_3 \perp e_4$, $e_4 \perp e_5$ and $e_5 \perp e_1$. With classical probabilities, orthogonal events are disjoint sets and the maximum for $\sum \mu(e_k)$ is 2 (Wright's inequality). This can easily be seen looking at Figure 1. The overlapping areas contribute twice to the sum which thus reaches its maximum when the probability is concentrated on these areas and all other areas carry zero probability; this maximum is 2.

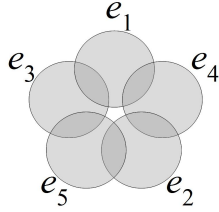


Figure 1. The classical case

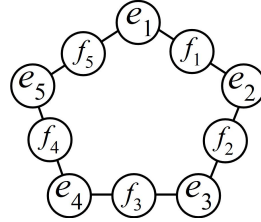


Figure 2. A quantum logic

In a quantum logic E , the situation is different. As an example, consider the quantum logic with the Greechie diagram shown in Figure 2. Each one of the five straight lines represents a Boolean algebra 2^3 . A state on this quantum logic is defined by $\mu(e_k) = 1/2$ and $\mu(f_k) = 0$ for $k = 1, \dots, 5$; this is Wright's pentagon state. Then $\sum \mu(e_k) = 5/2$. On the other hand, $e_k \perp e_{k+1}$ for $k = 1, \dots, 4$ and $e_5 \perp e_1$. Therefore $\mu(e_k) + \mu(e_{k+1}) \leq 1$ for $k = 1, \dots, 4$, $\mu(e_5) + \mu(e_1) \leq 1$ and then $2 \sum \mu(e_k) \leq 5$. This proves that $5/2$ is the maximum for $\sum \mu(e_k)$ in general quantum logics and that this maximum can only be achieved if $1 = \mu(e_5) + \mu(e_1) = \mu(e_k) + \mu(e_{k+1})$, $k = 1, \dots, 4$. The only state satisfying this is the Pentagon state above, since $\mu(e_1) = 1 - \mu(e_2) = \mu(e_3) = 1 - \mu(e_4) = \mu(e_5) = 1 - \mu(e_1)$ implies $\mu(e_k) = 1/2$ for $k = 1, \dots, 5$.

Applying the usual Hilbert space formalism, Wright showed that the pentagon state is impossible in quantum mechanics [19]. Now it will be seen that the calculus of conditional probability and the absence of third-order interference already suffice to rule out the pentagon state in a very broad probabilistic setting.

Theorem: $\sum \mu(e_k) \leq \frac{5}{2}$ if e_1, \dots, e_5 lie in any UCP quantum logic E with $I_3 \equiv 0$.

Proof. Assume that E is a UCP quantum logic with $I_3 \equiv 0$ and that $\sum \mu(e_k) = 5/2$. This is possible only when $\mu(e_k + e_{k+1}) = 1$ for $k = 1, \dots, 4$ as well as $\mu(e_1 + e_5) = 1$

(see above). Using the general properties of the maps T_e (section 2) as well as their orthogonal additivity in e implied by $I_3 = 0$ (section 3), it follows for any x in A and $k = 1, \dots, 4$ that $\mu(x) = \mu(T_{e_k+e_{k+1}}x) = \mu(T_{e_k}x) + \mu(T_{e_{k+1}}x)$ and $\mu(x) = \mu(T_{e_1+e_5}x) = \mu(T_{e_1}x) + \mu(T_{e_5}x)$. Subtracting each of these five identities from the next one results in $\mu(T_{e_{k-1}}x) = \mu(T_{e_{k+1}}x)$ for $k = 2, 3, 4$, $\mu(T_{e_5}x) = \mu(T_{e_2}x)$ and $\mu(T_{e_4}x) = \mu(T_{e_1}x)$. Hence, $\mu(T_{e_1}x) = \mu(T_{e_3}x) = \mu(T_5x) = \mu(T_{e_2}x) = \mu(T_{e_4}x)$. Finally, with $x = e_k$, $\mu(e_k) = \mu(T_{e_k}e_k) = \mu(T_{e_{k+1}}e_k) = 0$ for $k = 1, \dots, 4$ and $\mu(e_5) = \mu(T_{e_5}e_5) = \mu(T_{e_1}e_5) = 0$, which is a contradiction to $\sum \mu(e_k) = 5/2$. \square

For the study of contextuality, instead of $\sum \mu(e_k)$, Klyachko et al. [11] consider $K := \mu(\sum_{k=1}^4 x_k x_{k+1} + x_5 x_1)$ with the $\{+1, -1\}$ -valued observables $x_k := 2e_k - 1$. Then $x_k x_{k+1} = 1 - 2e_k - 2e_{k+1}$ and $K = 5 - 4 \sum \mu(e_k)$ [6]. It follows immediately that, under the assumptions of the theorem, K cannot reach its theoretical minimum -5 .

In the following section, a further simple consequence of the theorem concerning nonlocality and the Popescu-Rohrlich box will be considered.

5. The Popescu-Rohrlich box

For the study of nonlocality, four $\{+1, -1\}$ -valued observables a_1, a_2, b_1, b_2 are usually considered. The two observables a_1 and a_2 constitute a part of the system controlled by one party usually named Alice, and the observables b_1 and b_2 constitute a second part of the system controlled by another party usually named Bob. Motivated from the spatial separation of the two parts, it is assumed that the joint probability distribution p_{mn} of a_m and a_n exists for each $m, n = 1, 2$; $p_{mn}(r, s)$ is the probability of the measurement outcomes $a_m = r$ and $b_n = s$ for $r = \pm 1$, $s = \pm 1$. Relativistic causality requires the so-called *no-signaling principle*: $p_{m1}(r, +1) + p_{m1}(r, -1) = p_{m2}(r, +1) + p_{m2}(r, -1)$ and $p_{1n}(+1, s) + p_{1n}(-1, s) = p_{2n}(+1, s) + p_{2n}(-1, s)$ for $m, n = 1, 2$, $r, s = \pm 1$.

		a_1		a_2	
		+1	-1	+1	-1
b_1	+1	0	1/2	0	1/2
	-1	1/2	0	1/2	0
b_2	+1	0	1/2	1/2	0
	-1	1/2	0	0	1/2

Table 1. PR box

		a_1		a_2	
		+1	-1	+1	-1
b_1	+1	e_5			
	-1	e_1		e_4	
b_2	+1	e_2			
	-1			e_3	

Table 2. e_1, \dots, e_5

A measure for the statistical correlations between the two systems are the expectation values c_{mn} of the products $a_m b_n$ ($m, n = 1, 2$). The maximum for $|c_{11} + c_{12} + c_{21} - c_{22}|$ is 2 in the classical case (CHSH inequality [8]) and $2\sqrt{2}$ in quantum mechanics (Tsirelson's bound [18]). The general algebraic maximum is 4 and can be reached without violating the no-signaling principle only by the so-called Popescu-Rohrlich boxes or, briefly, PR boxes [15]. A PR box allocates specific probabilities to the four joint distributions p_{mn} , $m, n = 1, 2$; one is shown in in Table 1. There are

seven further PR boxes which can all be derived from the one in Table 1 by exchanging a_1 with a_2 , b_1 with b_2 , or $+1$ with -1 .

Again, the whole Hilbert space formalism of quantum mechanics is not necessary to rule out the PR boxes and the maximum 4 for $|c_{11} + c_{12} + c_{21} - c_{22}|$, but the calculus of conditional probability and the absence of third-order interference already suffice. This is an immediate consequence of the theorem in the last section when applied to the events e_1, \dots, e_5 as defined in Table 2 (see also [7]). Note that it is assumed that two events occurring in the PR box scenario are orthogonal when they involve different values for the same observable.

A related result has recently been presented in [14] where Tsirelson's bound is derived. This is a stronger result which, however, requires some further mathematical assumptions.

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